

Quadratic invariants for resonant clusters in discrete wave turbulence

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Abstract

We consider discrete clusters of exactly resonant triads arising from the Hamiltonian three-wave equation. A cluster consists of N modes forming a total of M connected triads. We investigate the problem of constructing a linearly independent set of quadratic constants of motion. We show that this problem is equivalent to an underlying basic linear problem, consisting of finding the null space of a rectangular $M \times N$ matrix A with entries 1, -1 and 0. In particular, we prove that the number of independent quadratic invariants is equal to $J \equiv N - M^* \geq N - M$, where M^* is the number of linearly independent rows in A . We formulate an algorithm for decomposing large clusters into smaller ones and show how various invariants are related to certain parts of a cluster, including the basic structures leading to $M^* < M$. We illustrate our findings by examples taken from the Charney-Hasegawa-Mima wave model. We also present a classification of small (up to four-triad) clusters.

1 Introduction

In this paper we will look at a system of weakly interacting waves where the leading order nonlinearity is quadratic. Examples include geophysical Rossby waves [1] and drift waves in plasma [2] both described by the Charney-Hasegawa-Mima (CHM) equation. Nonlinear interactions can be non-resonant, but the energy exchange is maximised when the waves are in resonance. Moreover, in the limit of very small amplitudes only the waves that are in exact resonance remain interacting. In this paper we will deal with such a limit of very weak waves. For quadratic nonlinearity these interactions take place between triplets of waves which form what is known as a resonant triad (provided three-wave resonance conditions, (1.1) below, can be satisfied). Let us work in d -dimensional Fourier space with wave vectors $\mathbf{k} \in \mathbb{R}^d$. A resonant triad is made up of three modes with wave vectors, \mathbf{k}_1 , \mathbf{k}_2 , \mathbf{k}_3 and frequencies, $\omega(\mathbf{k}_1)$, $\omega(\mathbf{k}_2)$, $\omega(\mathbf{k}_3)$ which satisfy the following three-wave resonance conditions:

$$\begin{aligned} \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2 &= 0, \\ \omega(\mathbf{k}_3) - \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2) &= 0. \end{aligned} \tag{1.1}$$

The wave vectors \mathbf{k} can either be continuous or discrete. For waves systems in an unbounded domain, the \mathbf{k} 's are continuous variables. Therefore, any \mathbf{k} may be a member of infinitely many resonant triads. However, in this paper we will look at wave systems in bounded domains where the wave vectors are discrete variables. For simplicity, let us consider waves in a d -periodic box with all sides being length $L = 2\pi$ and wave vectors $\mathbf{k} \in \mathbb{Z}^d$. As a result, any \mathbf{k} may now be a member of only a few resonant triads. Triads which are connected via common modes can be grouped together to form *resonant clusters* of various sizes ranging from “butterflies”, where two triads are joined via one mode, to a multiple-triad cluster involving a complicated network of interconnected triads. These clusters have been studied in [3, 4, 5, 6].

Types of wave turbulence

There are three different regimes of wave turbulence - kinetic, discrete and mesoscopic, and these have been classified in [7] and [8] by different relationships between the nonlinear frequency broadening Γ and the frequency spacing

$$\Delta\omega = \left| \frac{\partial\omega_{\mathbf{k}}}{\partial\mathbf{k}} \right| \frac{2\pi}{L} \sim \frac{\omega_{\mathbf{k}}}{kL}.$$

When wave amplitudes are very small, the nonlinear frequency broadening is much less than the frequency spacing:

$$\Gamma \ll \Delta\omega.$$

This is *discrete* wave turbulence and only waves that are in exact resonance can interact and exchange energy. Very large clusters are rare and there are usually a large number of small clusters, the simplest being an isolated triad. If the energy of the system is initially concentrated in these small clusters, then an energy cascade cannot take place. An extreme version of such a situation is when there are no resonant triads at all, like in the case of the capillary surface waves [9], in which case turbulence is “frozen”. For larger amplitudes, the nonlinear frequency broadening gets bigger and originally isolated clusters may become connected via quasi-resonances, that is, resonances with small enough frequency detuning. This will allow energy to be transferred between waves which are not exactly resonant, however, this is less efficient than energy transfer between waves which are in exact resonance.

If the wave system is forced weakly but continuously, the amplitudes will eventually become sufficiently large and the resonance broadening will become approximately the same size as the frequency spacing:

$$\Gamma \sim \Delta\omega.$$

In this region both types of wave turbulence - discrete and kinetic, exist and the system may oscillate in time between the two regimes giving rise to a new type of wave turbulence - *mesoscopic* wave turbulence, which is characterised by sand-pile behaviour (see [7] and [8]). This is because during the discrete phase the wave energy accumulates until the resonance broadening becomes of order of the frequency spacing and it is then released to higher k 's in the form of an “avalanche” with predominantly kinetic interactions. Until the moment of triggering the avalanche, the broadening increases. However, in the process of the avalanche release, the mean wave amplitude lowers so that the value of the broadening decreases again.

Thus, at this point the system returns to the energy accumulation stage in the discrete regime, and the cycle repeats.

For much larger levels of forcing the resonance broadening Γ will always greatly exceed $\Delta\omega$, in which case the wave system will be in the *kinetic* regime and an energy cascade between the forcing and dissipation scales gets triggered. In this paper, however, we will only deal with discrete, exact three-wave resonances.

The Hamiltonian

In order for us to write down the dynamical system for a discrete cluster let us first introduce the Hamiltonian equation in Fourier space:

$$i\dot{a}_{\mathbf{k}} = \frac{\delta\mathcal{H}}{\delta a_{\mathbf{k}}^*}, \quad (1.2)$$

where $a_{\mathbf{k}}$ is the amplitude of the Fourier mode corresponding to the wave vector \mathbf{k} , $*$ denotes the complex conjugate and the Hamiltonian \mathcal{H} is represented as an expansion in powers of $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^*$:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_2 + \mathcal{H}_3 + \dots, \\ \mathcal{H}_2 &= \sum_{\mathbf{k}} \omega_{\mathbf{k}} |a_{\mathbf{k}}|^2, \\ \mathcal{H}_3 &= \sum_{1,2,3} V_{12}^3 a_1 a_2 a_3^* \delta_{12}^3 + c.c. \end{aligned} \quad (1.3)$$

Here $a_j \equiv a_{\mathbf{k}_j}$ and $\delta_{12}^3 \equiv \delta(\mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2)$ is the Kronecker symbol which is one if $\mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2 = 0$ and zero otherwise. $V_{12}^3 \equiv V(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3)$ is the nonlinear interaction coefficient. \mathcal{H}_2 is the quadratic term and describes the non-interacting linear waves. \mathcal{H}_3 is the cubic term and describes the decaying of a single wave into two waves or the confluence of two waves into a single one. Here we will consider expansions up to and including \mathcal{H}_3 . Three-wave interactions dominate wave systems with small nonlinearity provided that $\mathcal{H}_3 \neq 0$ and the three-wave resonant conditions (1.1) are satisfied for a non-empty set of waves. Otherwise, the leading nonlinear processes may be four-wave interactions or even higher. Inserting \mathcal{H}_2 and \mathcal{H}_3 into equation (1.3) we have the evolution equation:

$$i\dot{a}_{\mathbf{k}} = \omega_{\mathbf{k}} a_{\mathbf{k}} + \sum_{1,2,\mathbf{k}} (V_{12}^{\mathbf{k}} a_1 a_2 \delta_{12}^{\mathbf{k}} + 2V_{\mathbf{k}2}^{1*} \delta_{\mathbf{k}2}^1 a_2 a_1^*). \quad (1.4)$$

In terms of an interaction representation variable, $b_{\mathbf{k}} = a_{\mathbf{k}} e^{i\omega_{\mathbf{k}} t}$, equation (1.4) can be rewritten as:

$$i\dot{b}_{\mathbf{k}} = \sum_{1,2,\mathbf{k}} (V_{12}^{\mathbf{k}} \delta_{12}^{\mathbf{k}} b_1 b_2 e^{-i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})t} + 2V_{\mathbf{k}2}^{1*} \delta_{\mathbf{k}2}^1 b_2 b_1^* e^{-i(\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}} - \omega_{\mathbf{k}_2})t}). \quad (1.5)$$

For very weak waves, the factors of $e^{-i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})t}$ in equation (1.5) oscillate rapidly leading to nearly zero time averages and therefore to a negligible cumulative effect of the respective terms. That is, however, unless modes \mathbf{k} , \mathbf{k}_1 and \mathbf{k}_2 are in exact resonance, in which case

$e^{-i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})t} = 1$. Leaving only such non-oscillating terms and neglecting all the non-resonant contributions amounts to replacing $e^{-i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})t}$ with $\delta(\omega_{12}^{\mathbf{k}})$ where $\omega_{12}^{\mathbf{k}} = \omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2}$. Thus equation (1.5) can be replaced with:

$$i\dot{b}_{\mathbf{k}} = \sum_{1,2,\mathbf{k}} (V_{12}^{\mathbf{k}} \delta_{12}^{\mathbf{k}} b_1 b_2 \delta(\omega_{12}^{\mathbf{k}}) + 2V_{\mathbf{k}2}^{1*} \delta_{\mathbf{k}2}^1 b_2 b_1^* \delta(\omega_{\mathbf{k}2}^1)). \quad (1.6)$$

This set of equations can be divided into independent subsets i.e. resonant clusters, and within each subset the waves interact among themselves but not with the waves of the other subsets. We can write the equations for the smallest cluster consisting of one resonant triad only:

$$\begin{aligned} \dot{b}_1 &= W^* b_2^* b_3, \\ \dot{b}_2 &= W^* b_1^* b_3, \\ \dot{b}_3 &= -W b_1 b_2, \end{aligned} \quad (1.7)$$

where $W = 2iV_{12}^3$. This triad is represented schematically in figure 1.

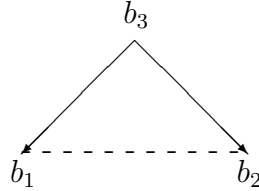


Figure 1: An isolated triad.

2 Linear systems of equations

Let us consider a number of resonant triads that are joined together forming a resonant cluster. Let the cluster consist of M triads and N modes, $b_n(t)$, $n = 1, \dots, N$. The resonant conditions for the m -th triad, defined by the equations (1.1) can be put into the following form:

$$\sum_{n=1}^N A_{mn} \mathbf{k}_n = \mathbf{0}, \quad \sum_{n=1}^N A_{mn} \omega_n = 0, \quad m \text{ fixed, } m = 1, \dots, M \quad (2.1)$$

where $\omega_n \equiv \omega_{\mathbf{k}_n}$, and for each fixed m the set $\{A_{mn}\}_{n=1}^N$ contains exactly two elements with value 1, one element with value -1 , and the remaining elements are equal to zero. In other words, the m -th row of the $M \times N$ matrix A corresponds to the resonant conditions for the m -th triad i.e.

$$\begin{aligned} A_{mn_1} \mathbf{k}_{n_1} + A_{mn_2} \mathbf{k}_{n_2} + A_{mn_3} \mathbf{k}_{n_3} &= \mathbf{0}, \\ A_{mn_1} \omega_{n_1} + A_{mn_2} \omega_{n_2} + A_{mn_3} \omega_{n_3} &= 0, \end{aligned}$$

where out of A_{mn_1}, A_{mn_2} and A_{mn_3} , two have value 1 and the third has value -1 . Let us from now on refer to matrix A as the *cluster matrix*.

Quadratic invariants

Some nonlinear problems are completely integrable - their behaviour is organised and regular; whereas non-integrable systems are not solvable exactly and exhibit chaotic behaviour. A $2n$ -dimensional Hamiltonian system is said to be classically integrable in the sense of Liouville if it admits n independent conserved quantities which are in involution (one can include \mathcal{H} among the conserved quantities). The following theorem allows us to find a conserved quantity I .

Theorem 1. *Let $\varphi_n \equiv \varphi_{\mathbf{k}_n}$ be a real function of the wavenumbers of the modes in the cluster and let φ_n satisfy the resonance condition $\sum_{n=1}^N A_{mn}\varphi_n = 0$ for all triads in the cluster. Then,*

$$I = \sum_{n=1}^N \varphi_n |b_n(t)|^2 = \text{const.}, \quad (2.2)$$

i.e. I is a quadratic invariant.

Proof. To show I is indeed an invariant, let us take the time derivative:

$$\frac{dI}{dt} = \sum_{n=1}^N \varphi_n (\dot{b}_n b_n^* + b_n \dot{b}_n^*). \quad (2.3)$$

Substitute for \dot{b}_n using equation (1.6) with $\tilde{V}_{12}^{\mathbf{k}} = V_{12}^{\mathbf{k}} \delta_{12}^{\mathbf{k}} \delta(\omega_{12}^{\mathbf{k}})$:

$$\begin{aligned} \dot{I} &= \sum_{n=1}^N \varphi_n b_n^* \sum_{1,2}^N (-i)(\tilde{V}_{12}^{\mathbf{k}} b_1 b_2 + 2\tilde{V}_{\mathbf{k}2}^{1*} b_2 b_1^*) + c.c \\ &= -i \sum_{1,2,3} (\varphi_3 b_3^* b_1 b_2 \tilde{V}_{12}^3 + \varphi_1 b_1^* \tilde{V}_{12}^{3*} b_2^* b_3 + \varphi_2 b_2^* \tilde{V}_{12}^{3*} b_1^* b_3 \\ &\quad - \varphi_3 b_3 b_1^* b_2^* \tilde{V}_{12}^{3*} - \varphi_1 b_1 b_2 b_3^* \tilde{V}_{12}^3 - \varphi_2 b_2 b_1 b_3^* \tilde{V}_{12}^3) \\ &= -i \sum_{1,2,3} (\varphi_3 - \varphi_1 - \varphi_2)(b_3^* b_1 b_2 \tilde{V}_{12}^3 + c.c). \end{aligned}$$

It is clear that $\dot{I} = 0$ if $\varphi_3 - \varphi_1 - \varphi_2 = 0$ for every term in the sum, i.e. for every resonant triad. \square

This is similar to the conservation laws for the three-wave kinetic equation found in [10] which is valid for kinetic wave turbulence discussed in section 6.

By theorem 1, for an isolated triad described by the dynamical system (1.7) above I takes the form:

$$I = \varphi_1 |b_1|^2 + \varphi_2 |b_2|^2 + \varphi_3 |b_3|^2.$$

And it turns out that there are two independent quadratic integrals of motion called Manley-Rowe invariants:

$$\begin{aligned} I_{13} &= |b_1|^2 + |b_3|^2, \\ I_{23} &= |b_2|^2 + |b_3|^2. \end{aligned} \tag{2.4}$$

Here, the resonant condition $\varphi_3 - \varphi_1 - \varphi_2 = 0$ is clearly satisfied when $\varphi_1 = \varphi_3 = 1$, $\varphi_2 = 0$ and $\varphi_2 = \varphi_3 = 1$, $\varphi_1 = 0$ respectively. It is obvious that a triad is integrable and the exchange of energy between the modes is periodic. Since dynamical system (1.7) has complex amplitudes $b_n(t)$, there are six variables, the real and imaginary parts of each, so for a triad to be integrable three conserved quantities are needed, namely the Hamiltonian and invariants (2.4). However, larger clusters may not be integrable and the motion is no longer regular but chaotic.

Let us discuss some examples of the quadratic invariants. The most well known ones are the energy and the momentum components with their density φ_n equal to ω_n and the components of the wave vector $\mathbf{k} = (k_x, k_y)$ respectively. Remarkably, in the system of Rossby/drift waves one other example of φ_n satisfying the resonant conditions is already known and has been discussed in the literature in the context of kinetic wave turbulence [12, 13, 14, 7]. This quadratic invariant is called *zonostrophy* and it will be discussed in this paper when we consider examples from the Rossby/drift wave systems later in sections 4 and 6.

According to theorem 1 the problem of finding quadratic invariants amounts to finding the null space of the rectangular $M \times N$ cluster matrix A . The null space of a matrix A is the set of all vectors $\boldsymbol{\varphi}^{(n)} \equiv (\varphi_1^{(n)}, \varphi_2^{(n)}, \dots, \varphi_N^{(n)})^T$ for which

$$A\boldsymbol{\varphi}^{(n)} = \mathbf{0}. \tag{2.5}$$

So each solution $\boldsymbol{\varphi}$ of equation system (2.5) gives (and is given by) a quadratic invariant of the original system of ODEs (1.7). Let us define Φ as a matrix, the columns of which are linearly independent basis vectors $\boldsymbol{\varphi}$, which span the null space of A and thereby they correspond to a complete set of linearly independent quadratic invariants. We will call Φ the *null space matrix*. Examples of finding quadratic invariants and respective the null space matrices will be shown in section 3.

2.1 Properties of a cluster and its matrix A

1. From the matrix A , the total number of independent quadratic invariants (solutions for $\boldsymbol{\varphi}^{(n)}$ in equation (2.5)) is equal to:

$$J \equiv N - M^* \geq N - M,$$

where M^* is the number of linearly independent rows in A .

2. In order for the triads to be connected into a cluster, the following obvious condition must be satisfied:

$$2M + 1 \geq N.$$

For example, consider the triple-chain in figure 2 below. If $2M + 1 < N$, then N must be greater than 7. The only way to achieve this without adding a fourth triad to the cluster is to disconnect a triad from the chain.

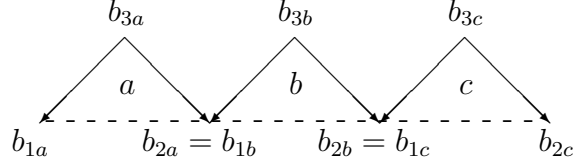


Figure 2: A triple chain.

3. By construction, each row of A consists of only three non-zero components, with values from the set $\{1, 1, -1\}$. Notice that the total number of vectors that can be constructed in such a way is equal to:

$$N(N - 1)/2.$$

Thus,

$$M \leq N(N - 1)/2.$$

4. Let $D (\leq d)$ be the dimension of the polyhedron formed by the cluster's modes in the d -dimensional wave vector space. Then the total number of invariants:

$$J = N - M^* \geq D.$$

To see this, notice that the equations for the cluster's wave vectors are:

$$\sum_{n=1}^N A_{mn} \mathbf{k}_n = \mathbf{0}, \quad (2.6)$$

and we know there are at most J independent solutions of the scalar equation $\sum_{n=1}^N A_{mn} \varphi_n =$

0. Therefore when we look at the Cartesian components of equations (2.6), we see that there are at most J independent solutions. In the case $J \geq d$, the best scenario would be one independent solution for each component, or $D = d$, so $J \geq D$. In the case $J < d$, the best scenario is $D = J$ and any other scenario gives $D < J$.

Consider $d = 2$ and require that resonant modes do not lie on the same line, so $D = 2$. Then $N - M^* \geq 2$, so that the solution sets of equations (2.6), $\{k_{x,n}\}_{n=1}^N$ and $\{k_{y,n}\}_{n=1}^N$ are allowed to be linearly independent.

Now let $d = 3$ and require that resonant modes are not in the same plane, so $D = 3$. Then $N - M^* \geq 3$, so that the solution sets of equations (2.6), $\{k_{x,n}\}_{n=1}^N$, $\{k_{y,n}\}_{n=1}^N$ and $\{k_{z,n}\}_{n=1}^N$ can be linearly independent.

Notice the obvious fact that for an isolated triad we have $N - M^* = 2$, so the result is $2 \geq D$ i.e. for any host dimension d , the isolated triad lies either on a plane or a line.

5. If for a cluster $\{A_m\}_{m=1}^M$ a mode belongs to only one triad, say triad number m' then the row $A_{m'}$ corresponding to that triad is linearly independent of the other rows in A . This is because the column corresponding to such a mode will be non-zero (1 or -1) in row $A_{m'}$ only.

2.2 Physical requirements and excluded cluster matrices

We present three physical requirements on the cluster matrices and their null spaces, so that the clusters represent physically sensible sets of interacting modes. Namely, by writing out the resonant conditions for each triad from equations (2.6), one must admit only the matrices A_{mn} for which the solution set of wavenumbers \mathbf{k}_n , $n = 1, \dots, N$, is physically sensible.

1. The first physical requirement is that in the solution of equations (2.6), no two wavevectors are equal (i.e., $\mathbf{k}_n \neq \mathbf{k}_{n'}$ if $n \neq n'$).

Mathematically, this requirement is summarised in the statement: We will exclude a cluster matrix A_{mn} if its null space is orthogonal to any of the vectors $\mathbf{e}_i - \mathbf{e}_j$, for some $i, j = 1, \dots, N$, where \mathbf{e}_i is the canonical basis vector with components $(\mathbf{e}_i)_k = \delta_{ik}$, $i, k = 1, \dots, N$.

For example, any two rows $A_m, A_{m'}$ with $m \neq m'$, must not have the same values in more than one column. The reason being that two rows having equal values in two columns would imply that the corresponding column of the third wave vector should be equal, so the two vectors would represent exactly the same triad. Therefore the following matrices are not physically sensible:

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 &= 0, \\ \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_4 &= 0, \end{aligned}$$

$$\implies \mathbf{k}_3 = \mathbf{k}_4.$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 &= 0, \\ \mathbf{k}_1 - \mathbf{k}_3 + \mathbf{k}_4 &= 0, \end{aligned}$$

$$\implies \mathbf{k}_2 = \mathbf{k}_4.$$

2. Second, if the underlying PDE is for a real function, so that there is an identification between the wavevectors \mathbf{k} and $-\mathbf{k}$, then an extra requirement is that in the solution of equations (2.6), no two wavevectors add up to zero (i.e., $\mathbf{k}_n \neq -\mathbf{k}_{n'}$ if $n \neq n'$). In fact, since $b_{-\mathbf{k}} = b_{\mathbf{k}}^*$ one can work with only half of the \mathbf{k} -space for convenience, in which case possibilities to have simultaneously \mathbf{k} and $-\mathbf{k}$ are automatically excluded. Mathematically, this requirement is summarised in the statement: We will exclude a cluster matrix A_{mn} if its null space is orthogonal to any of the following vectors: $\mathbf{e}_i + \mathbf{e}_j$, for some $i, j = 1, \dots, N$.

For example, any two rows $A_m, A_{m'}$ with $m \neq m'$, must not have values of opposite sign in two components (for example, (1,-1) for row A_1 and (-1,1) for row A_2). If this was to happen the corresponding \mathbf{k} of the third non-zero component of row A_m should be equal to minus the \mathbf{k} of the third non-zero component of row $A_{m'}$, so one of the two vectors would be outside the half of the \mathbf{k} -space, whichever way this half is selected to describe wave fields which are real in the physical space.

Therefore the following matrix is not physically sensible:

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix} \quad \begin{aligned} \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 &= 0, \\ \mathbf{k}_3 + \mathbf{k}_4 - \mathbf{k}_1 &= 0, \end{aligned}$$

$$\implies \mathbf{k}_2 = -\mathbf{k}_4.$$

3. Third, in cases when the zero-mode must be excluded, one must require that in the solution of equations (2.6), no wavevector is the zero vector (i.e., $\mathbf{k}_n \neq \mathbf{0}$ for all n).

Mathematically, this requirement is summarised in the statement: We will exclude a cluster matrix A_{mn} if its null space is orthogonal to any of the following vectors: \mathbf{e}_i , for some $i = 1, \dots, N$.

We remark that in the case of triad interactions, the violation of the third requirement will imply the violation of either the first or the second requirement for some modes. To see this, notice that, for example, if $\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 = \mathbf{0}$ and $\mathbf{k}_3 = \mathbf{0}$, then $\mathbf{k}_1 = -\mathbf{k}_2$.

As an example of this third case, we have an excluded type of cluster which has the shape of a tetrahedron (see figure 16). This has the following cluster matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \quad \begin{aligned} \mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_4 &= 0, \\ -\mathbf{k}_1 + \mathbf{k}_3 + \mathbf{k}_4 &= 0, \\ \mathbf{k}_2 - \mathbf{k}_3 + \mathbf{k}_4 &= 0, \end{aligned}$$

$$\implies \mathbf{k}_4 = 0$$

$$\implies \mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3.$$

See the discussion around figure 16 for more details about this unphysical case.

We remark that it is possible to construct another type of tetrahedron cluster which violates the second requirement only: this type of cluster, while not useful for the CHM model (because in CHM the underlying fields are real), could be useful in theories for complex fields. An example of this cluster is given by the following cluster matrix:

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \end{bmatrix} \quad \begin{aligned} \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 &= 0, \\ \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4 &= 0, \\ -\mathbf{k}_1 + \mathbf{k}_3 + \mathbf{k}_4 &= 0, \end{aligned}$$

$$\implies \mathbf{k}_4 = -\mathbf{k}_2.$$

From here on, we will consider only clusters that satisfy the three physical requirements outlined above.

3 Classification of clusters

In this section we will describe clusters up to and including triple-triad clusters. More complete list of such clusters, their matrices and invariants can be found in appendix A. We begin with the basic building block for all clusters - an isolated triad.

3.1 An isolated triad

The simplest dynamical system in the case of three wave resonances is a system of three modes, b_1, b_2 and b_3 , called an isolated triad.

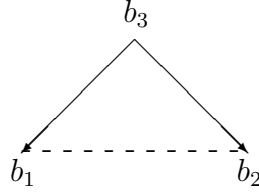


Figure 3: An isolated triad.

It's dynamical system reads:

$$\begin{aligned}\dot{b}_1 &= W^* b_2^* b_3, \\ \dot{b}_2 &= W^* b_1^* b_3, \\ \dot{b}_3 &= -W b_1 b_2,\end{aligned}\tag{3.1}$$

where W is the interaction coefficient. The cluster matrix corresponding to the resonant conditions for an isolated triad is:

$$A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}.$$

Its null space matrix, the set of vectors $\varphi^{(n)}$ for which $A\varphi^{(n)} = \mathbf{0}$ is:

$$\Phi = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A triad has $N - M = 2$ independent quadratic invariants in Manley-Rowe form. So each column of Φ gives a quadratic invariant of the dynamical system. They are:

$$\begin{aligned}I_1 &= |b_2|^2 - |b_1|^2, \\ I_2 &= |b_1|^2 + |b_3|^2.\end{aligned}\tag{3.2}$$

A linear combination of these is also a quadratic invariant:

$$I_3 = I_1 + I_2 = |b_2|^2 + |b_3|^2.$$

Any two of these quadratic invariants are linearly independent and therefore a triad is integrable in the sense of Liouville.

Note, that in a triad there are two different types of modes, a P for passive mode and an A for active mode. b_1 and b_2 are P modes and b_3 is the A mode. There is only one active mode in each triad. They correspond to substantially different scenarios of energy flux among the modes and this is discussed in [3, 5, 6].

3.2 Double-triad clusters

Two triads can be joined together to form a double-triad cluster.

3.2.1 Butterflies

Two triads, a and b, can be connected via one mode to form a butterfly. There are three different types of connection: PP, AP or AA.

A PP-butterfly consists of two triads a and b, connected via mode $b_{1a} = b_{1b}$, which is passive in both triads.

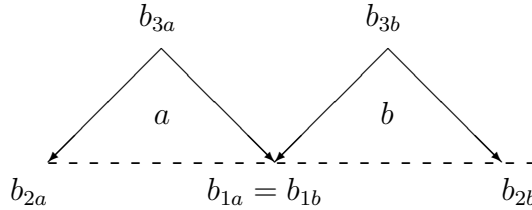


Figure 4: A PP-butterfly.

It's dynamical system reads:

$$\begin{aligned} \dot{b}_{1a} &= W_a^* b_{2a}^* b_{3a} + W_b^* b_{2b}^* b_{3b}, \\ \dot{b}_{2a} &= W_a^* b_{1a}^* b_{3a}, \\ \dot{b}_{2b} &= W_b^* b_{1a}^* b_{3b}, \\ \dot{b}_{3a} &= -W_a b_{1a} b_{2a}, \\ \dot{b}_{3b} &= -W_b b_{1a} b_{2b}. \end{aligned} \tag{3.3}$$

The cluster matrix is:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

and the null space matrix is:

$$\Phi = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It has $N - M = 3$ independent quadratic invariants of the form:

$$\begin{aligned} I_1 &= |b_{2a}|^2 + |b_{3a}|^2, \\ I_2 &= |b_{2a}|^2 + |b_{2b}|^2 - |b_{1a}|^2, \\ I_3 &= |b_{1a}|^2 - |b_{2a}|^2 + |b_{3b}|^2. \end{aligned} \tag{3.4}$$

These can be linearly combined to find more quadratic invariants:

$$\begin{aligned} I_4 &= |b_{2b}|^2 + |b_{3b}|^2, \\ I_5 &= |b_{1a}|^2 + |b_{3a}|^2 + |b_{3b}|^2. \end{aligned}$$

However, only three of the five quadratic invariants above are linearly independent and therefore a butterfly is not integrable in the sense of Liouville since five conserved quantities including the Hamiltonian are needed.

Here we constructed the dynamical system for the butterfly by writing out the dynamical system for each triad a and b and substituting for the common mode, i.e. $b_{1a} = b_{1b}$. Where the two triads meet via a common mode the corresponding right hand sides are summed. This rule applies also to bigger clusters. However, from now on we will not write out the dynamical equations explicitly. Firstly, there is no need to do so since such dynamical equations can easily be reproduced from the cluster matrix A . And secondly, for our purpose of finding invariants the cluster matrix is a more straightforward and self-sufficient approach. Likewise we will omit writing out the explicit expressions for the invariants as they can easily be produced using the null space matrix of A . Tables containing A and Φ for each of the clusters listed below can be found in appendix A.

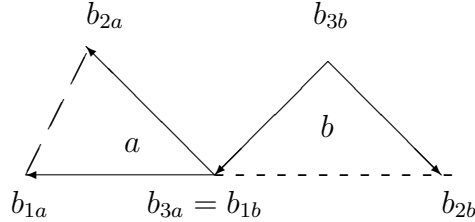


Figure 5: An AP-butterfly.

From now on we will omit the dashed lines from the clusters in order to make the figures less busy. The triads can easily be identified by labelling them a, b, c etc., and the arrows point out off the only active mode in the triad.

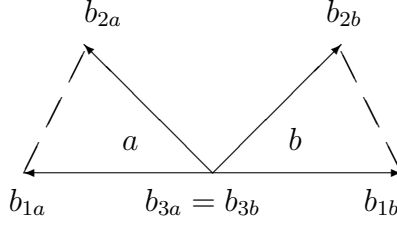


Figure 6: An AA-butterfly.

3.2.2 Kites

Another way of joining two triads is via two common modes, to form what is known as a kite. There is only one possible way in which to do this and that is as an AP-PP kite connected via modes $b_{3a} = b_{1b}$ and $b_{2a} = b_{2b}$ as in figure 7:

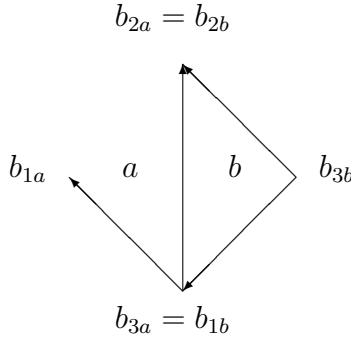


Figure 7: An AP-PP kite.

The corresponding cluster matrix is:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}.$$

It has $N - M = 2$ independent quadratic invariants and the null space matrix is:

$$\Phi = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We will now explain why certain types of kites are not realisable. Firstly, it is obvious that there cannot be a kite with the connection AA-AA, or even AP-AP, since there cannot be two active modes in one triad. Now consider a kite with the connection PP-PP; this can be shown to be wrong by considering the resonant condition, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ for both triads:

$$\begin{aligned}\mathbf{k}_{1a} + \mathbf{k}_{2a} &= \mathbf{k}_{3a}, \\ \mathbf{k}_{1b} + \mathbf{k}_{2b} &= \mathbf{k}_{3b}.\end{aligned}$$

Substituting $1a = 1b$ and $2a = 2b$, we find that:

$$\mathbf{k}_{3a} = \mathbf{k}_{3b}.$$

This is not possible since we would be left with, $b_{1a} = b_{1b}$, $b_{2a} = b_{2b}$ and $b_{3a} = b_{3b}$ which means a and b are the same triad. The same happens for a AA-PP kite, with connecting modes $b_{1a} = b_{1b}$ and $b_{3a} = b_{3b}$. Finally let's consider a kite with an AP-PA connection. Substituting $3a = 1b$ and $1a = 3b$ we find that:

$$\mathbf{k}_{2a} = -\mathbf{k}_{2b}.$$

If the underlying wave field is real in the physical space, as is the case for Rossby waves and drift waves (see examples in section 4) then \mathbf{k} and $-\mathbf{k}$ represent the same mode via $b_{-\mathbf{k}} = b_{\mathbf{k}}^*$ and therefore the two triads in the kite are identical. Thus the AP-PA kite is impossible for real wave fields.

3.3 Triple-triad clusters

Triple-triad clusters consist of three triads, denoted here as a, b and c. There are stars with one common mode, chains with two common modes and triangles with three common modes; see figures 8, 9 and 10 respectively. We will present a list of the cluster matrices A and the null space matrices Φ in appendix A. There also exist a number of other ways to join together three triads in a cluster, shown in appendix A.1, which don't contradict the dynamical rules considered above in section 2.

3.3.1 Stars

Three triads, a, b and c, can be connected via one mode to form a star. There are four different types of connection: AAA, AAP, PPA or PPP.

3.3.2 Chains

Triple-chains are three triads joined together by two modes. There are seven types of connections: PP-PP, AP-PP, PA-PP, AP-PA, PA-PA, AA-PP, AA-PA.

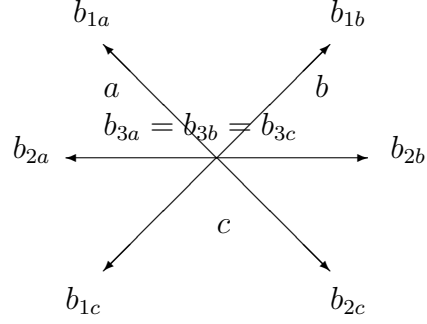


Figure 8: An AAA-star.

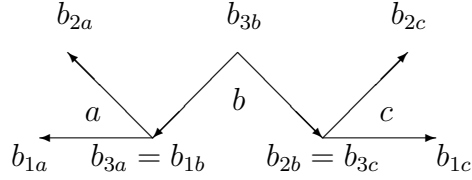


Figure 9: An AP-PA chain.

3.3.3 Triangles

Triple-triangles have three common modes in three triads. There are seven types: PP-PP-PP, PA-PP-PP, PA-PA-PP, AP-PA-PP, AA-PP-PP, AA-PA-PP and AP-AP-AP.

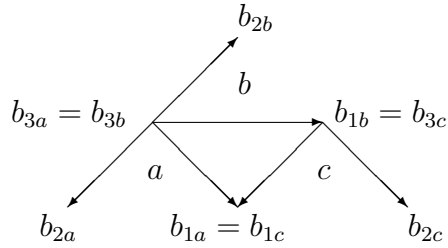


Figure 3.8:

Figure 10: An AA-PA-PP triangle.

4 Examples of clusters: Charney-Hasegawa-Mima model

Charney-Hasegawa-Mima (CHM) model describes geophysical Rossby waves and waves in magnetized plasmas. It is defined by the following equation for a real wave field ψ in two dimensional physical space:

$$\frac{\partial}{\partial t}(\Delta\psi - F\psi) + \beta\frac{\partial\psi}{\partial x} + J[\psi, \Delta\psi] = 0, \quad (4.1)$$

where $F = 1/\rho^2$ with ρ being the Rossby deformation radius or the ion Larmor radius for Rossby and drift waves respectively and β is the gradient of the Coriolis parameter or a measure of density gradient in plasma.

According to the CHM model (4.1) the dispersion relation for the wave frequency is given by:

$$\omega_{\mathbf{k}} = \frac{-\beta k_x}{F + k^2}. \quad (4.2)$$

Since ψ is a real function then \mathbf{k} and $-\mathbf{k}$ represent the same mode via the property of the Fourier transform of real functions $b_{-\mathbf{k}} = b_{\mathbf{k}}^*$. Thus we can choose to work with only half of the Fourier space e.g. $k_x \geq 0$. We will further neglect $k_x = 0$ since this corresponds to zero frequency zonal flows and these are not waves.

Let us consider examples of clusters found in a specified region of \mathbf{k} -space in two dimensions, k_x, k_y in the limits of the very small and very large F .

Example 1: Let the frequency be that of small-scale Rossby waves, $\rho k \rightarrow \infty$. The dispersion relation can be obtained by putting $F = 0$ in equation (4.2) which gives:

$$\omega_{\mathbf{k}} = \frac{-\beta k_x}{k^2}. \quad (4.3)$$

If we consider the region: $1 \leq k_x \leq 100$ and $-100 \leq k_y \leq 100$ we find a total of thirty-four clusters (seventeen clusters plus their mirror images). This consists of twenty-four isolated triads, four butterflies, two triple-chains, two seven-triad clusters and two thirteen-triad clusters; see figure 11.

Example 2: Now let us consider large scale Rossby waves, $\rho^2 k^2 \ll 1$, with the frequency obtained from equation (4.2) by Taylor expansion in $\rho^2 k^2$:

$$\omega_{\mathbf{k}} = -\beta k_x \rho^2 (1 - \rho^2 k^2). \quad (4.4)$$

Since the first part in this expression is equal to \mathbf{k}_x , for the purpose of finding the resonances we can take a simpler expression:

$$\omega_k = k_x k^2. \quad (4.5)$$

If we consider the region $1 \leq k_x \leq 20$ and $-20 \leq k_y \leq 20$ we find a total of four clusters. This consists of two isolated triads, one ten-triad cluster and one 104-triad cluster as shown in figure 12.

We see that in a much smaller domain of the large-scale system we already have a much larger cluster than in the small-scale system. This tells us that the resonance conditions are much easier to satisfy in the large-scale limit than in the small-scale limit.

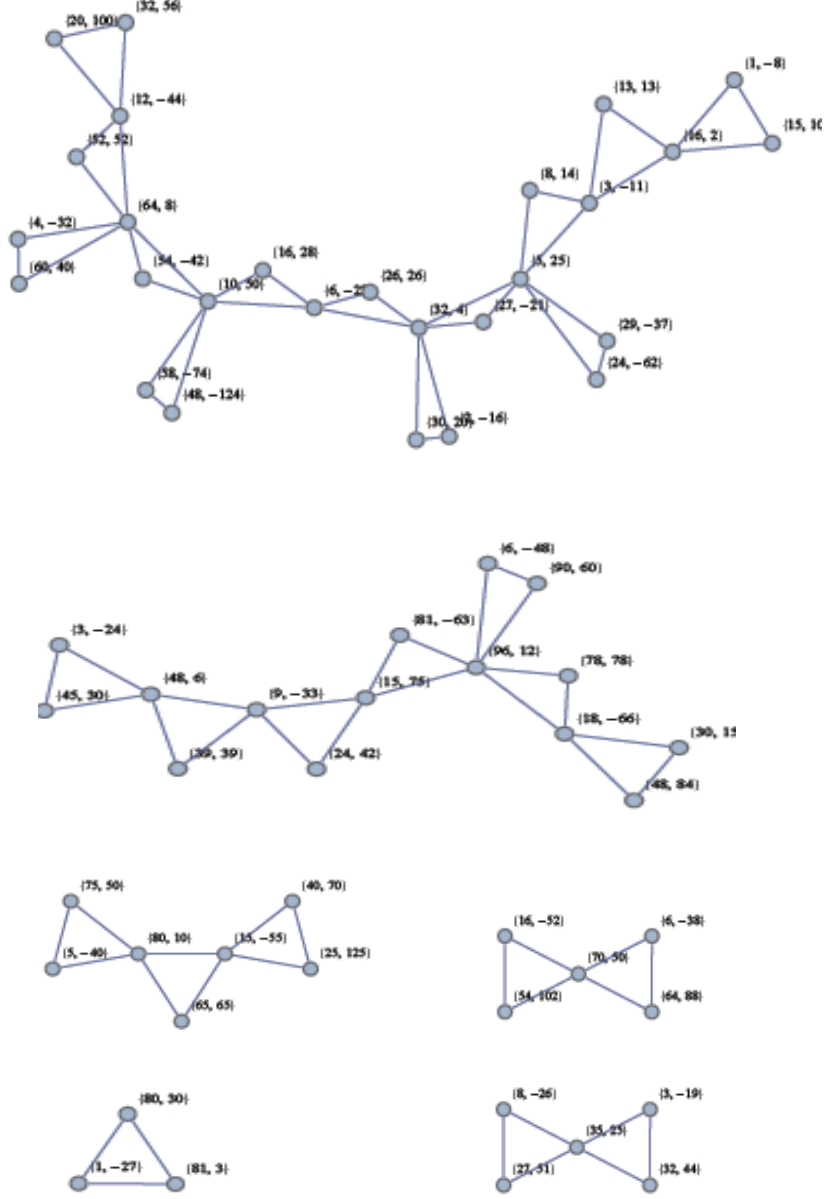


Figure 11: Small-scale Rossby waves in the region $1 \leq k_x \leq 100$, $-100 \leq k_y \leq 100$. The mirror images have been removed and only one isolated triad is shown.

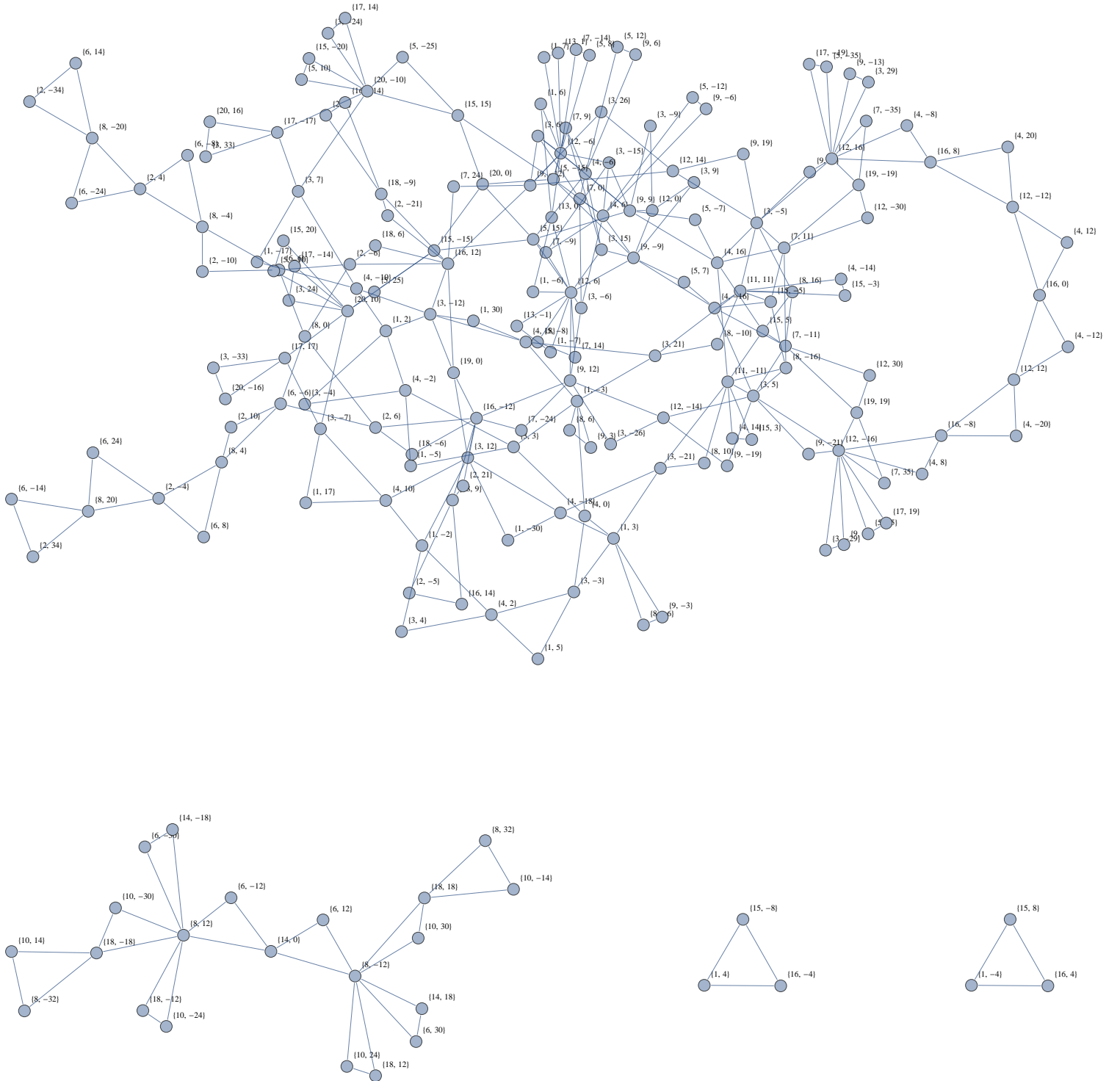


Figure 12: Large-scale Rossby waves in the region $1 \leq k_x \leq 20$, $-20 \leq k_y \leq 20$.

5 Reducing clusters

In this section we develop an algorithm for decomposing larger clusters into smaller clusters which completely determine the properties of the original larger clusters. This will allow us to associate particular quadratic invariants with small subsets of modes of the original big cluster.

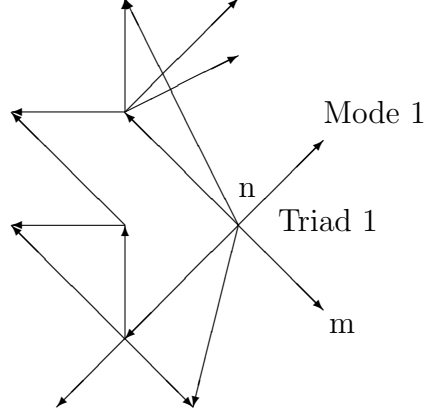


Figure 13: A cluster before reduction.

Part 1

- Consider a cluster, like in figure 13, with an unconnected mode, let us call it mode 1 for simplicity. Being unconnected, means that mode 1 will not be joined to any other triad in the cluster other than the one it belongs to, call it triad 1.
- Consider the cluster matrix A . Since mode 1 is unconnected, the rest of column 1 in the cluster matrix will contain zeros.
- Delete column/mode one and row/triad one to leave a new reduced matrix A' :

$$A = \begin{matrix} & \begin{matrix} 1 & & n & & m \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 & \dots \\ 0 & . & . & . & . & . & . \\ 0 & . & & & & & \\ 0 & . & & A' & & & \\ 0 & . & & & & & \\ 0 & . & & & & & \end{bmatrix} \end{matrix}.$$

- Consider a vector from the null space of A (a column of matrix Φ):

$$\begin{matrix} & 1 & & n & & m & & \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 & \dots \\ 0 & . & . & . & . & . & . \\ 0 & . & & & & & \\ 0 & . & & & A' & & \\ 0 & . & & & & & \\ 0 & . & & & & & \end{bmatrix} & \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ . \\ \varphi_n \\ . \\ \varphi_m \\ . \\ . \\ . \\ \varphi_N \end{bmatrix} & = 0. \end{matrix}$$

Here n and m denote the positions of the non-zero entries in row one (other than the first position).

- Solve $A' \begin{bmatrix} \varphi_2 \\ \vdots \\ \varphi_N \end{bmatrix} = 0$, to find the null space matrix of A' .
- Then solving for φ_1 we have:

$$\varphi_1 - \varphi_n + \varphi_m = 0 \longrightarrow \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ . \\ . \\ . \\ \varphi_N \end{bmatrix} = \begin{bmatrix} \varphi_n - \varphi_m \\ \varphi_2 \\ \vdots \\ \varphi_N \end{bmatrix}. \quad (5.1)$$

Thus finding the null space matrix of A is reduced to finding the null space matrix of a smaller matrix A' . By eliminating one row which is linearly independent from the rest of the rows in A , and eliminating the respective column, we have not changed the null space dimension. Therefore, the number of independent invariants is the same for A and the smaller matrix A' .

- Two situations may arise:
 1. Triad 1 has one unconnected mode. A' is a cluster matrix of a cluster obtained from A by eliminating triad 1 only. It is clear that such reduced clusters will have the same number of invariants as the original bigger cluster.
 2. Triad 1 has two unconnected modes e.g. 1 and m in the example below:

$$A = \begin{matrix} & \begin{matrix} 1 & & n & & m \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 & \dots \\ 0 & . & . & . & . & 0 & . \\ 0 & . & & & & 0 & \\ 0 & . & & A' & & 0 & \\ 0 & . & & & & 0 & \\ 0 & . & & & & 0 & \end{bmatrix} \end{matrix}.$$

In this case we have a column of zeros in matrix A' (column m). Thus, one can now choose φ_m arbitrarily as follows,

$$\begin{bmatrix} \varphi_1 = \varphi_n - \varphi_m \\ \varphi_2 \\ \vdots \\ \varphi_m \\ \vdots \\ \varphi_N \end{bmatrix} = \begin{bmatrix} \varphi_n \\ \varphi_2 \\ \vdots \\ 0 \\ \vdots \\ \varphi_N \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}. \quad (5.2)$$

The second contribution here, $[-1, 0, \dots, 1, \dots, 0]^T$, gives us one of the linearly independent invariants of A , and all the other columns in the null space matrix of A must have zero entries in the position number m . Note that this invariant is an attribute solely of the triad with two loose ends which we are eliminating, and in fact it has a simple Manley-Rowe form, $I = |b_m|^2 - |b_1|^2$.

In the example above we eliminated a triad with two loose ends of type P (the ones at which the arrows point). Another possibility is when one loose end is of P type and the other is of A type (the one from which the arrows originated). It is easy to see that in this case the respective extra invariant is also of a Manley-Rowe type, $I = |b_m|^2 + |b_1|^2$.

- Note that by removing triads with unconnected modes you may possibly disjoin the remaining cluster into independent clusters, which must then be treated separately.
- Repeat until you are left with a fully connected cluster(s) i.e. one in which all modes are connected to more than one triad.

If you have completed part 1 and are left with the matrix A' that cannot be reduced any further via this method move to part 2. Note however, that some clusters can be fully decomposed by repeating part 1 only and part 2 will not be necessary. This is the case for all the clusters given in the example of the small-scale Rossby waves above (figure 11). In particular the largest (thirteen-triad cluster) has six triads with two loose ends each yielding a total of six Manley-Rowe type invariants as explained above. These triads are eliminated in the first application of part 1 after which a seven-triad chain remains which will be further reduced by successive elimination of triads with double loose ends.

Part 2

Suppose in the remaining cluster there are two triads (triad 1 and 2 in figure 14 below) that are joined together by two modes and that these modes are not connected to any other triad.

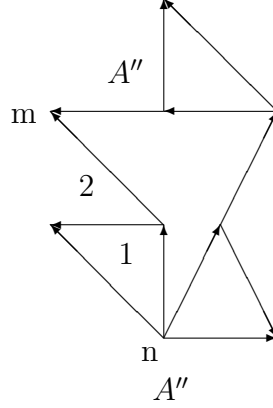


Figure 14: A cluster after all loose ends have been removed by part 1.

- Rearrange the rows/columns of A' to form a 2×2 matrix in the top left hand corner (i.e. renumber the modes in the triad in an appropriate way). The only 2×2 matrix possible is $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ or any permutation of it because it must satisfy the exclusion principles meaning the following matrices are not allowed: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Triads 1 and 2 will form a PP-PA kite.
- Delete the pair of modes in column one and two to get matrix A'' as follows:

$$A' = \begin{bmatrix} & & & n & m & & \\ 1 & 1 & 0 & -1 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & . & . & . & . & . \\ 0 & 0 & . & & & & \\ 0 & 0 & . & & A'' & & \\ 0 & 0 & . & & & & \end{bmatrix}.$$

- By removing a pair of connected triads you may disjoin the remaining cluster into two independent clusters, in which case each must be treated separately. Or the remaining part may stay as a single cluster.

- Solve $A'' \begin{bmatrix} \varphi_3 \\ \vdots \\ \varphi_N \end{bmatrix} = 0$, to find the null space matrix of A'' .

- Then solving for φ_1 and φ_2 we have:
 $\varphi_1 + \varphi_2 - \varphi_n = 0$ and $\varphi_1 - \varphi_2 + \varphi_m = 0$

$$\rightarrow \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \cdot \\ \cdot \\ \cdot \\ \varphi_N \end{bmatrix} = \begin{bmatrix} 1/2(\varphi_n - \varphi_m) \\ 1/2(\varphi_n + \varphi_m) \\ \varphi_3 \\ \vdots \\ \varphi_N \end{bmatrix}. \quad (5.3)$$

Therefore, the null space of A is uniquely constructed from the null space of A'' and has the same dimension. In other words, by eliminating two triads as described above, it leads to a smaller cluster (or two disjoint clusters) whose total number of independent invariants is equal to the number of independent invariants in the original cluster.

It is not possible to have any zero columns in A'' since these should have been eliminated in part 1. The necessity to remove zero columns may arise only at the level of eliminating single triads and not at the level of triad pairs.

- Look at A'' (single or two disjoint clusters) and again search for unconnected single modes (part 1) and triad pairs (part 2) of the type: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Repeat the procedure until part 1 and part 2 cannot be applied any more.

Part 3

After a number of successive applications of part 1 and part 2 one inevitably arrives at reduced cluster(s) for which the steps of neither part 1 nor part 2 can be carried out. Such reduced cluster(s) are usually significantly smaller than the original one but it will still have the same number of invariants. Moreover the invariants for the big cluster can be easily reconstructed from the respective invariants of such a reduced cluster by expressing the entries of the eliminated modes in the null space matrix as shown in equations (5.1), (5.2) and (5.3). Because of the fact that this small cluster will completely determine the conservation properties of the entire original large cluster we will call it the cluster kernel of the original cluster. Note that not all clusters will have kernels as they can be taken apart completely by the steps of part 1 and part 2.

- Let us now consider a cluster kernel (irreducible by parts 1 and 2). Following the logic of part 1 and part 2 let us now consider 3×3 blocks in the top left hand corner (arising after appropriate renumbering of the rows/triads and columns/modes) such that the

rest of the entries below the 3×3 block contains zeros only, e.g.

$$A'' = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & -1 & 0 & \dots \\ 0 & 0 & 0 & . & . & . & . \\ 0 & 0 & 0 & . & & A''' & \\ 0 & 0 & 0 & . & & & \end{bmatrix}.$$

Let us call these 3×3 blocks $A^{3 \times 3}$. Of course $A^{3 \times 3}$ must again satisfy the exclusion principles discussed above. Either $A^{3 \times 3}$ has:

1. a non-zero determinant, in which case the system of equations for $\varphi_1, \varphi_2, \varphi_3$ has a unique solution and consequently the complete system has the same number of invariants as A''' .
2. or a zero determinant, in which case, one or two independent solutions (when the $A^{3 \times 3}$ rank is two or one respectively) are to be obtained by putting $\varphi_4 = \dots = \varphi_N = 0$. Further solutions are to be sought for $(\varphi_4, \dots, \varphi_N)$ given by solutions of $A'''(\varphi_4, \dots, \varphi_N)^T = 0$. For each of these solutions, the resulting system of equations for $\varphi_1, \varphi_2, \varphi_3$ has either:
 - (a) a unique solution or
 - (b) no solutions at all.

By Rouché-Capelli theorem [ref], case (2a) occurs when the rank of the coefficient matrix, $A^{3 \times 3}$, in the system of equations for $\varphi_1, \varphi_2, \varphi_3$ is equal to the rank of the augmented matrix, $[A^{3 \times 3}|\mathbf{b}]$ (where \mathbf{b} depends on the fixed values of $\varphi_4, \dots, \varphi_N$). Otherwise, if the rank of $A^{3 \times 3}$ is less than the rank of $[A^{3 \times 3}|\mathbf{b}]$ we will have (2b) i.e. no solutions.

Situation 2 is new with respect to 1×1 and 2×2 matrix eliminations above, since only starting at the 3×3 matrix level can we get degenerate matrices.

In case (2a) the system A'' has more invariants than A''' . Note that the value of $N - M$ is the same for matrix A''' as for the original matrix A because the size of A''' is less than the size of A by the equal amount of rows and columns. This means that in case (2a) the number of linearly independent rows in the original matrix A , M^* is less than the total number of rows M , i.e. the number of independent invariants of the full system is: $J = N - M^* > N - M$. An example of (2a) can be found in the following cluster:

It has the cluster matrix:

$$A'' = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 \end{bmatrix}.$$

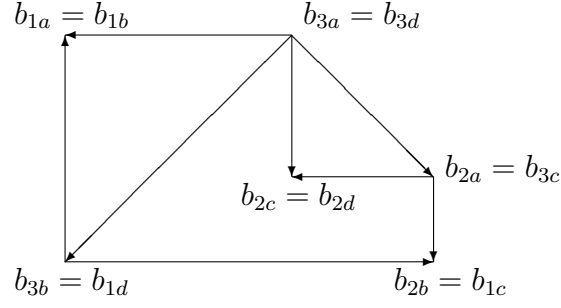


Figure 15: A cluster demonstrating case (2a) where $M^* < M$.

This cluster is fully connected and cannot be reduced by part 1 or part 2. Rearrange A'' to form a 3×3 matrix, $A^{3 \times 3}$, in the top left corner:

$$A'' = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & 1 & 1 \end{array} \right].$$

The determinant of $A^{3 \times 3}$ is zero, so let us use the Rouché-Capelli theorem. The rank of $A^{3 \times 3}$ is two. To apply the theorem we must find the vector \mathbf{b} . From A''' (the lower-right part in A'') it can be seen that $\varphi_4 = \varphi_5 + \varphi_6$. So we have two independent solutions: either $\varphi_4 = \varphi_5 = 1, \varphi_6 = 0$ or $\varphi_4 = \varphi_6 = 1, \varphi_5 = 0$.

To find \mathbf{b} corresponding to these solutions, multiply the rectangular matrix in the top-right of $A^{3 \times 3}$ in A'' by $(\varphi_4, \varphi_5, \varphi_6)^T$. So either:

$$\mathbf{b}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix},$$

or

$$\mathbf{b}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Using these in the augmented matrix $[A^{3 \times 3} | \mathbf{b}]$ it can be seen that the rank is again two. By the Rouché-Capelli theorem, this system of equations has an infinite number of solutions. Hence, the number of linearly independent rows $M^* = 3$ is less than the number of rows $M = 4$ and hence A has an extra invariant.

This explains why the null space matrix for A , $\Phi = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, doesn't satisfy the rule $J = N - M = 2$ but instead $J = N - M^* = 3$ i.e. $J = N - M^* \geq N - M$.

In case (2b) we “loose” solutions, i.e. we may find that extra solutions gained by solving for $\varphi_1, \varphi_2, \varphi_3$ with $\varphi_4 = \dots = \varphi_N = 0$ may be compensated by an equal or larger loss because some solutions $(\varphi_4, \dots, \varphi_N)$ of $A'''(\varphi_4, \dots, \varphi_N)^T = 0$ do not correspond to any solution of the full system A . Therefore in case (2b) we may have the number of independent solutions of the original cluster A to be the same or less than A''' . To see this take the following “tetrahedron cluster”:

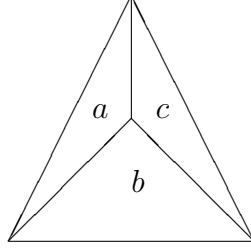


Figure 16: A tetrahedron cluster.

This cluster is unphysical (since the resulting null space is such that $\mathbf{k}_4 = \mathbf{0}$, thus violating the third physical requirement in Section 2.2) but for illustrating case (2b) it is a simple example to consider.

The cluster matrix which has been rearranged to form a 3×3 matrix, $A^{3 \times 3}$, on the left hand side and a column, \mathbf{b} , on the right is:

$$A'' = \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{array} \right].$$

The determinant of $A^{3 \times 3}$ is zero. Now in order to solve our system of equations:

$$\left[\begin{array}{cccc} 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{array} \right] \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} = 0,$$

let us form an augmented matrix: $[A^{3 \times 3} | \mathbf{b}]$ with $\mathbf{b} = (\varphi_4, \varphi_4, \varphi_4)$, i.e. we have:

$$A^{3 \times 3} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} + \varphi_4 \mathbf{b} = 0.$$

Since there is no A''' in this case, we can choose φ_4 arbitrarily e.g. $\varphi_4 = 1$. The rank of the coefficient matrix $A^{3 \times 3}$ is two and the rank of the augmented matrix $[A^{3 \times 3} | \mathbf{b}]$ is three. Hence, by the Rouché-Capelli theorem no solutions exist.

Now let $\varphi_4 = 0$ and solve $A^{3 \times 3}(\varphi_1, \varphi_2, \varphi_3)^T = 0$. Since the rank of $A^{3 \times 3}$ is two, we have one independent solution, $(\varphi_1, \varphi_2, \varphi_3) = (1, 1, 1)$.

Consequently, the tetrahedron cluster has $N - M = 1$ invariant and its null space matrix is:

$$\Phi = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore, for the tetrahedron cluster the number of invariants corresponds to the “ $N - M$ ” rule (which holds for non-degenerate cases) even though its $A^{3 \times 3}$ matrix is degenerate.

Let us now precede with examples of clusters that appear in the real systems. Consider the cluster taken from the large-scale CHM example above with $M = 104$ and $N = 178$ (figure 12). For this cluster the application of the algorithm amounted to carrying out part 1 three times which resulted in two cluster kernels, both made up of eight triads and twelve modes as shown in figures 17 and 18 (in this case it turns out that part 2 was not needed). Note that each of these clusters are mirror symmetric i.e. each cluster maps onto itself when transformation $\mathbf{k}_y \rightarrow -\mathbf{k}_y$ is applied. The fact that both clusters are the same size is interesting but probably coincidental.

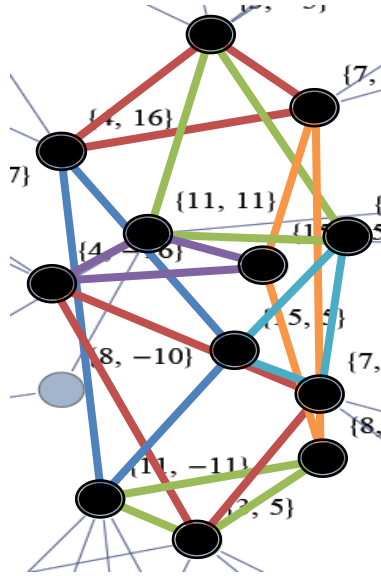


Figure 17: The first cluster kernel taken from figure 12 such that each triad is connected to other triads and neither part 1 or part 2 can be applied.

When applying our algorithm further to each of these clusters (figure 17 and 18) no 3×3 matrices can be formed in A'' such that the rest of the entries below it are zero. Likewise no 4×4 blocks can be formed. Consequently we are required to look for 5×5 blocks. It turns out that these 5×5 matrices, call them $A^{5 \times 5}$ can be found for both clusters (see appendix C) and both have zero determinants. For 5×5 matrices it is the same as for 3×3 matrices

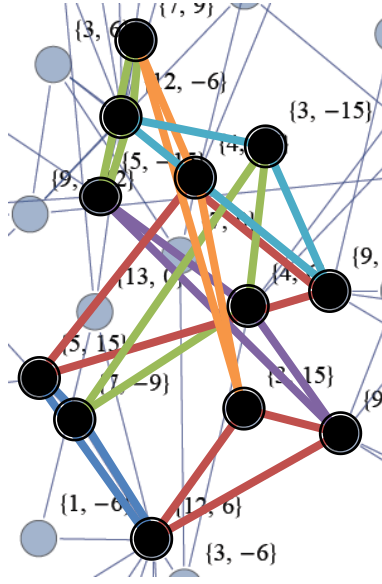


Figure 18: The second cluster kernel taken from figure 12.

in that there may be extra invariants if they are degenerate. Again we can now apply the Rouché-Capelli theorem to check whether there are solutions or no solutions. In fact both cluster kernels correspond to case (2a) where matrix $A^{5 \times 5}$ is degenerate and the rank of the coefficient matrix $A^{5 \times 5}$ is the same as the rank of the augmented matrix $[A^{5 \times 5} | \mathbf{b}]$ and hence there are an infinite number of solutions. As the rank of $A^{5 \times 5}$ is four which is one less than the size, this means that for each of these clusters the null space basis contains one extra vector (see appendix C) and so the 104-triad cluster will contain two extra invariants in total i.e. the number of independent quadratic invariants is equal to $J = N - M^* = N - M + 2$.

It is interesting to apply our algorithm to the simple two- and three-triad clusters even though only part 1 of the algorithm will be needed and obviously, there will be no remaining cluster kernels. For completeness let us start with a triad for which the algorithm is trivial. We know a triad has two linearly independent Manley-Rowe invariants (2.4). On the other hand you can also select two other independent invariants for the triad, for example, energy and one of the momentum components i.e. its x -component, which have density $\varphi_n = \omega_n$ and $\varphi_n = k_x$ respectively (see section 6).

Now take a butterfly of any of the three kinds, the first triad always has two loose ends and so contributes one invariant of the Manley-Rowe type. Once this triad is removed a single triad remains which contributes two further invariants. Hence a butterfly has three invariants in total. Cluster matrices corresponding to different types of butterflies and the null space matrices are given in appendix A. Like for the triad, one can choose the physical invariants, for example the energy and the two components of momentum. Also in the case of the CHM wave model one can choose the zonostrophy invariant as one of the independent invariants in the set of three. For more detail on zonostrophy see section 6.

Now consider a kite. The first triad only has one loose end so contributes no invariants. Once this triad is removed one triad remains which again contributes two invariants. Hence a kite has two invariants in total and again we can choose two physical invariants such as the energy and one of the momentum components.

It is also easy to apply our algorithm to triple-triad clusters: stars, chains and triangles. In all of these cases all triads are eliminated by three successive part 1 steps. These will have four, four and three invariants respectively, see appendix A. The most interesting new feature here arises in the case with four independent invariants. In 3D cases the energy and three components of momentum can be chosen as the set of four independent invariants. However, in the 2D cases the energy and the two components of momentum make only three invariants and one more physical invariant is needed. Such an extra invariant, zonostrophy, can be found in the CHM model i.e. the energy, the two momentum components and the zonostrophy make a full set of four independent invariants in the dynamics of the respective three-triad clusters: stars and chains.

The other three-triad clusters given in appendix A.1 are also taken apart by two part one steps and similarly, the four-triad clusters given in appendix ?? are taken apart by three part one steps. However, the four-triad cluster example discussed earlier in this section is irreducible by parts 1 and 2 and is a cluster kernel.

Part 1 of our algorithm is the easiest to apply since it amounts to finding the number of triads with two loose ends in a cluster. Part 2 is also fairly simple, but part 3 can become more complicated especially as the cluster size increases. Part 3, however, was useful to prove that there can be extra invariants and how these arise when the determinant of the 3×3 matrix is zero (as in the case of our 4×6 example). For bigger clusters a 3×3 matrix may no longer be found and instead we may have to move to 4×4 or 5×5 matrices (like our 104-triad cluster example) or higher levels of the algorithm. Although in principle the procedure for the higher levels is the same as for the 3×3 case, the calculations become more time consuming which reduces the advantage of carrying out these higher-order steps with respect to using alternative methods of finding the null space.

6 Physical invariants: the energy, momentum and zonostrophy

In this section we are going to consider the role of four physical invariants, the energy, momentum and zonostrophy, which belong to the class of quadratic invariants considered in this paper. We are also going to compare situations arising in our discrete regime to those in the kinetic regime. Firstly let us introduce briefly the kinetic regime.

The kinetic regime

The kinetic regime occurs when $\Gamma \gg \Delta\omega$, which is the opposite of the discrete regime, and is described by the kinetic equation (see [10] and [12]):

$$\dot{n}_{\mathbf{k}} = 4\pi \int |V_{12}^{\mathbf{k}}|^2 \delta_{12}^{\mathbf{k}} \delta(\omega_{12}^{\mathbf{k}}) \times [n_{\mathbf{k}_1} n_{\mathbf{k}_2} - n_{\mathbf{k}} n_{\mathbf{k}_1} \text{sign}(w_{\mathbf{k}} w_{\mathbf{k}_2}) - n_{\mathbf{k}} n_{\mathbf{k}_2} \text{sign}(w_{\mathbf{k}} w_{\mathbf{k}_1})] d\mathbf{k}_1 d\mathbf{k}_2, \quad (6.1)$$

where $n_{\mathbf{k}} = \epsilon_{\mathbf{k}}/\omega_{\mathbf{k}}$ is the wave action spectrum, $V_{12}^{\mathbf{k}}$ is the nonlinear interaction coefficient and $\epsilon_{\mathbf{k}}$ is the energy spectrum. It can be written in a symmetric form, since $k_x > 0$:

$$\dot{n}_{\mathbf{k}} = \int (R_{12k} - R_{k12} - R_{2k1}) d\mathbf{k}_1 d\mathbf{k}_2, \quad (6.2)$$

where

$$R_{12k} = 2\pi |V_{12}^{\mathbf{k}}|^2 \delta_{12}^{\mathbf{k}} \delta(\omega_{12}^{\mathbf{k}}) (n_{\mathbf{k}_1} n_{\mathbf{k}_2} - n_{\mathbf{k}} n_{\mathbf{k}_1} - n_{\mathbf{k}} n_{\mathbf{k}_2}).$$

Generally, for any quantity:

$$\Phi = \int \varphi_{\mathbf{k}} \dot{n}_{\mathbf{k}} d\mathbf{k},$$

with density $\varphi_{\mathbf{k}}$, it is conserved if:

$$\varphi_{\mathbf{k}_3} - \varphi_{\mathbf{k}_1} - \varphi_{\mathbf{k}_2} = 0.$$

In other words, there is an extra invariant if the resonant relation for the density is satisfied. This is the same as for discrete wave turbulence ($\Gamma \ll \Delta\omega$) even though they are very different regimes. Note that there is an intermediate regime known as mesoscopic wave turbulence in which $\Gamma \sim \Delta\omega$. This regime is much more complicated and consequently no results about additional invariants are known for this.

The invariants

Well-known examples of invariants are the energy and momentum with densities $\omega_{\mathbf{k}}$ and \mathbf{k} respectively, and for a generic wave system no other invariant besides these has been found. However, it was discovered in [12] for kinetic wave turbulence that one extra conserved quantity, independent of the energy and momentum, exists for the system of Rossby waves. This quantity is conserved under the same conditions as the kinetic equation, namely weak nonlinearity and random phases, and it cannot be conserved in interactions of higher order so may be called the invariant of the three systems. This invariant proved to be a unique additional invariant and thus the first example of wave systems with a finite number of additional invariants was obtained. This extra invariant is now known as zonostrophy.

The general expression for the density of zonostrophy, $\varsigma_{\mathbf{k}}$ was found for all \mathbf{k} 's in [13], it is:

$$\varsigma_{\mathbf{k}} = \arctan \frac{k_y + k_x \sqrt{3}}{\rho k^2} - \arctan \frac{k_y - k_x \sqrt{3}}{\rho k^2}. \quad (6.3)$$

In our paper we considered two limits- the small-scale and the large-scale limit. In [14] and [15] the limit $\rho k \rightarrow \infty$ was taken in the general expression to get the density in the case of small-scale turbulence:

$$\varsigma_{\mathbf{k}} = - \lim_{\rho \rightarrow \infty} \frac{5\rho^5}{8\sqrt{3}} (\rho_k - 2\sqrt{3}\omega/\beta\rho) = \frac{k_x^3}{k^{10}} (k_x^2 + 5k_y^2). \quad (6.4)$$

And if we take the large-scale limit in the general expression for zonostrophy (6.3) above we get:

$$\varsigma_{\mathbf{k}} = \frac{k_x^3}{k_y^2 - 3k_x^2}. \quad (6.5)$$

Let us consider how the energy, momentum and zonestrophy appear in our discrete clusters starting with the smallest. A triad has two linearly independent quadratic invariants (Manley-Rowe) and as a result the energy ($\omega_{\mathbf{k}}$), the two components of momentum (k_x, k_y) and the zonestrophy ($\varsigma_{\mathbf{k}}$) will not be linearly independent of one another. Only two may be linearly independent e.g. k_x and $\omega_{\mathbf{k}}$ or k_y and $\varsigma_{\mathbf{k}}$ etc.

To see this consider the Manley-Rowe equations:

$$\begin{aligned} I_1 &= |b_2|^2 - |b_1|^2, \\ I_2 &= |b_1|^2 + |b_3|^2, \end{aligned}$$

$$I = \varphi_1 |b_1|^2 + \varphi_2 |b_2|^2 + \varphi_3 |b_3|^2,$$

Substituting in $\varphi_3 = \varphi_1 + \varphi_2$ we get:

$$\begin{aligned} I &= \varphi_1 (|b_2|^2 + |b_3|^2) + \varphi_2 (|b_2|^2 + |b_3|^2), \\ &= \varphi_1 I_1 + \varphi_2 I_2. \end{aligned}$$

Now take a butterfly (two-triad cluster) which has three invariants in total. As a result zonestrophy does not appear as an extra invariant to the energy, and momentum components. However, any three of the four invariants will be linearly independent. Actually these considerations for a triad and a butterfly are general for all sizes. Consider larger clusters such as the triple-triad chains and stars, which both have four invariants, the zonestrophy in these cases does appear as an extra invariant as all four of $k_x, k_y, \omega_{\mathbf{k}}$ and $\varsigma_{\mathbf{k}}$ are linearly independent of one another.

Lets now consider bigger clusters arising from specific examples in the small and large scale limits. Take the biggest cluster found in the small-scale limit, shown in the top left corner of figure 11, which is made up of thirteen triads and twenty-seven modes and has fourteen linearly independent invariants. The cluster matrix A is shown in appendix B from which it can be seen that triad one (row one) has two loose ends (indicated via bold print). From the null space cluster, also shown in appendix B, it is clear that triad one (column one) has a Manley-Rowe invariant. Like wise for triads 4, 6, 9, 11 and 13.

Now let us consider the energy, momentum and zonestrophy invariants in more detail in relation to the thirteen triad cluster above. Substitute the coordinates k_x, k_y for each of the twenty-seven modes into the right hand side of equation (6.4) to find the values of the zonestrophy, $\varsigma_{\mathbf{k}}$. The x and y momentum are simply the values of k_x and k_y and to get the energy values substitute k_x and k_y into:

$$\omega_{\mathbf{k}} = \frac{k_x}{k_x^2 + k_y^2}.$$

Firstly, check that $\omega_{\mathbf{k}}$, k_x and $\varsigma_{\mathbf{k}}$ are in the null space of A and therefore are indeed invariants i.e. check that $A\omega_{\mathbf{k}} = 0$, $Ak_x = 0$ and $A\varsigma_{\mathbf{k}} = 0$. Let us now represent each of the invariants $\omega_{\mathbf{k}}$, k_x and $\varsigma_{\mathbf{k}}$ as linear combinations of basis vectors that span the null space of A . To do this we must find coefficient matrices a , b and c such that:

$$\omega = \Phi a, k_x = \Phi b, \varsigma = \Phi c.$$

Using Matlab to solve the above and rounding to three decimal places we have:

$$a = [0.046, 0.038, 0.031, 0.005, 0.023, 0.023, 0.019, 0.015, 0.003, 0.012, 0.004, 0.010, 0.008, 0.002]^T,$$

$$b = [15, 13, 8, 24, 27, 30, 26, 16, 48, 54, 4, 52, 32, 20]^T,$$

$$c = 1.0e-006*[0.675, 0.505, 0.450, 0.000, 0.026, 0.021, 0.016, 0.014, 0.000, 0.001, 0.000, 0.001, 0.000, 0.000]^T.$$

From c this it is clear that the first three vectors contain the most zonostrophy and by looking at A and Φ in appendix B it can be seen that the first three triads in-fact contain most of the zonostrophy. This is not surprising since they have the smallest wave vectors, \mathbf{k} . Again, to a slightly lesser extent, it can be seen from a the first three vectors also contain the most energy.

For completion let us now take an example from the large scale limit. We will consider the cluster made up of ten-triads and twenty-one modes in the bottom left corner of figure 12. The cluster matrix A and the null space matrix Φ are both listed in appendix B. This time to find the zonostrophy values we must substitute k_x and k_y into equation (6.5). Now find the coefficient matrix c such that $\varsigma = \Phi c$:

$$c = [-9.615, 1.667, -7.043, -10.394, 6.000, 6.000, -7.043, -10.394, 1.667, 0.615, -9.615]^T.$$

From c it is interesting to notice that the negative values in rows 1,3,4,7,8 and 11 coincide with the columns containing Manley-Rowe invariants in the null space matrix Φ .

7 Summary and Conclusions

In this paper we consider weak, dispersive waves involved in three-wave resonant interactions, i.e. a system of waves with quadratic nonlinearity which satisfy the 3-wave resonant conditions (1.1) for some of the modes. We consider discrete wave turbulence where the nonlinear frequency broadening is much less than the frequency spacing between adjacent wave modes, $\Gamma \ll \Delta\omega$. (The opposite regime where $\Gamma \gg \Delta\omega$ corresponds to kinetic wave turbulence, see [10]). We prove a theorem relating quadratic invariants and the wave resonance relations. It turns out that although discrete and kinetic wave turbulence are different physical regimes, the conservation conditions appear to be very similar: the \mathbf{k} -space density of the quadratic invariant must satisfy the same resonance conditions as do the wave vector and the frequency.

As wave turbulence is discrete, the resonant manifold splits into a set of resonant clusters of finite size ranging from individual triads to much larger multiple-triad clusters. Each cluster evolves independently of the others, and therefore conservation properties hold independently for each one. In such a case of a finite dimensional cluster the resonant condition for the density of the invariant is reduced to a linear algebra system of equations. Namely, the problem of finding invariants can be reformulated as finding the null space of the cluster matrix A introduced in section 2 whose horizontal dimension N is given by the number of modes and the vertical dimension M is given by the number of triads.

We present a classification of smaller clusters and their conservation properties up to and including three-triad clusters. We give specific examples that arise in both the small- and

large-scale limits of the Charney-Hasegawa-Mima (CHM) equation. We introduce a general algorithm for finding quadratic invariants for large clusters via a step-by-step reduction of such clusters to smaller blocks. This algorithm allows us (at the top level) to identify local invariants associated with individual triads independent of the rest (triads with two loose ends). At each step of this algorithm invariants of larger clusters are associated with and expressed in terms of invariants of a smaller reduced cluster. The algorithm allows us to prove the existence of the cases where the number of independent invariants is larger than $N - M$ and to give examples of these cases, explaining how these situations are related to the degeneracy of smaller blocks within the matrix (the smallest being 3×3). We illustrate our algorithm by applying it to a large 104-triad cluster arising in the large-scale CHM system, and show that it has $N - M + 2$ invariants. We also discuss the role of well known physical invariants e.g. energy, momentum and the extra invariant in Rossby wave systems, zonostrophy, in the context of our discrete clusters.

Even though in this paper we only considered three-wave systems, generalisation to the four-wave and to the higher-order wave systems is straightforward. In future, it would be interesting to consider such higher-order systems, eg. clusters of linked quartets in the system of deep water gravity waves.

It is also straightforward to generalise our approach to the wave clusters which involve frequency-detuned triads. Recall that quasi-resonant triads become important as the level of wave turbulence rises. In fact, our Theorem 1 will remain almost identical for such clusters: the density of each quadratic invariant must satisfy exact condition of resonance for each triad (even though the frequency is no longer in exact resonance). For detuned clusters, we must leave factors $e^{-i(\omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})t}$ in equation (1.5) (ie. refrain from replacing them with $\delta(\omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})$). The subsequent working for finding a quadratic invariant comprises in following the steps of proving Theorem 1. Expressions $e^{-i(\omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})t}$ will factor out and will not modify the resulting resonant condition on the invariant's density.

Obviously, the quasi-resonant clusters will contain more triads than the respective resonant clusters. Thus, as the detuning increases, $N - M$ decreases, and generally there will be less invariants. In particular, if in a cluster all triads are connected via two common modes, then the number $N - M$ is equal to 2, corresponding to energy and enstrophy conservation (see [16]). In future, it would be interesting to study the dynamical consequences of such a loss of the quadratic invariants when the frequency detuning is growing due to an increase of the wave turbulence strength. In general, it would also be interesting to simulate numerically wave turbulence in large discrete clusters, resonant or quasi-resonant, conservative or forced-dissipated, to see how their behaviour is different from their counterparts in kinetic wave turbulence, and how presence of numerous additional quadratic invariant affects the turbulent cascades.

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A Appendix: Two- and three-triad clusters

For reference, we present a table of the two- and three-cluster matrices and the respective null spaces.

Type of cluster	Connection	Invariants	Cluster matrix - A	Null space matrix - Φ
AP-butterfly	$3a = 1b$	3	$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
AA-butterfly	$3a = 3b$	3	$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
AAA-star	$3a = 3b = 3c$	4	$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
AAP-star	$3a = 3b = 1c$	4	$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
PPA-star	$1a = 1b = 3c$	4	$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
PPP-star	$1a = 1b = 1c$	4	$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Type of cluster	Connection	Invariants	Cluster matrix - A	Null space - Φ
PP-PP-PP triangle	$2a = 1b, 2b = 2c, 1a = 1c$	3	$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
PA-PP-PP triangle	$2a = 3b, 2b = 2c, 1a = 1c$	3	$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
PA-PA-PP triangle	$2a = 3b, 2b = 3c, 1a = 1c$	3	$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
AP-PA-PP triangle	$3a = 1b, 2b = 3c, 1a = 1c$	3	$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
AA-PP-PP triangle	$3a = 3b, 2b = 2c, 1a = 1c$	3	$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
AA-PA-PP triangle	$3a = 3b, 1b = 3c, 1a = 1c$	3	$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
AP-AP-AP triangle	$3a = 2b, 3b = 2c, 1a = 3c$	3	$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Table A.1: A table to show the cluster matrix, null space and number of invariants for the clusters classified in section 3.

A.1 Additional three-triad clusters

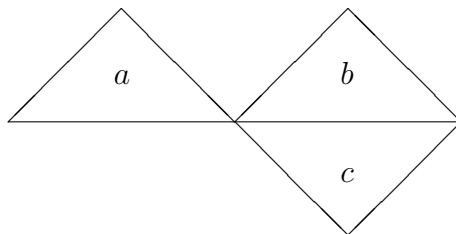


Figure 19: Two common modes.

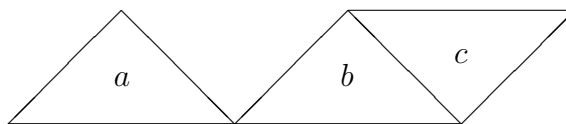


Figure 20: Three common modes.

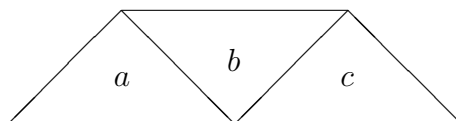


Figure 21: A trapezium - three common modes.

B Appendix: Large cluster arising in the small-scale CHM model

Let us consider the largest cluster found in the small-scale limit of the CHM model pictured in figure 11. It consists of thirteen linked triads all of which can be eliminated by successively applying part 1 of our reduction algorithm.

For the thirteen-triad cluster the cluster matrix is:

[illegible]

Its null space matrix is:

[illegible]

We can immediately see five invariants which involve only two waves each. As we mentioned in the main text, they correspond to Manley-Rowe type of invariants belonging locally to each of the triads with two loose ends each. But on the graph of the thirteen-triad cluster we see six of such triads! Easy to see that the sixth Manly-Rowe invariant is just the sum of the last two columns in the null space matrix.

Thus we see the considerable number invariant in this example, that six out of fourteen, are local to their respective triads. Their existence is probably imposing severe restrictions on moving energy in and out of these triads and propagating them throughout the cluster. The other invariants involve more waves and as such they are more global. We could expect that they also play an important role in stirring the energy through the \mathbf{k} -space in a way it happens the continuous case (kinetic wave turbulence) due to the presence of the zonostrophy invariant [12, 14]. In the latter case, presence of the zonostrophy causes anisotropy of the energy cascade which results in creation of large-scale zonal flows. Effect of the quadratic invariant onto the discrete turbulence in clusters is an interesting subject for further studies.

C Appendix: Cluster kernels of the 104-triad cluster in the large-scale CHM model

Let us consider the giant 104-triad cluster (“frog”) of the large-scale CHM model shown in figure 12. As we said in the main text, by three successive part-1 steps of our reduction algorithm this cluster can be reduced to the cluster kernels shown in figures 17 and 18. Both of these kernels appear to be so tightly linked that no further reduction is possible by removing triad pairs (kites), triple- or even four-triad blocks. This brings us straight to considering 5×5 blocks.

Figure 17: A'' has been rearranged to form a 5×5 matrix in the top left hand corner:

$$\left[\begin{array}{ccccc|ccccc} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

The determinant of $A^{5 \times 5}$ is zero and the rank is four. Now find the vector \mathbf{b} from A''' :

$$-\varphi_7 + \varphi_9 + \varphi_{11} = 0,$$

$$\varphi_8 - \varphi_{11} + \varphi_{12} = 0,$$

$$-\varphi_6 + \varphi_{10} + \varphi_{12} = 0.$$

One independent solution is $\varphi_7 = \varphi_8 = \varphi_9 = \varphi_{10} = 1$ and $\varphi_{12} = -1$ and $\varphi_6 = \varphi_{11} = 0$. So

$$\mathbf{b} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

The rank of $[A^{5 \times 5} | \mathbf{b}]$ is four. So by the Rouché-Capelli theorem the cluster kernel in figure 17 has an infinite number of solutions and since the rank of the coefficient matrix is one less than its size, one extra invariant. The null space for figure 17 is:

$$\Phi = \begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Figure 18: Once again A'' has been rearranged to form a 5×5 matrix in the top left hand corner:

$$\left[\begin{array}{ccccc|cccccc} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \end{array} \right]$$

The determinant of $A^{5 \times 5}$ is zero and the rank is four. Now find the vector \mathbf{b} from A''' :

$$\varphi_{10} + \varphi_{11} - \varphi_{12} = 0,$$

$$\varphi_7 + \varphi_8 - \varphi_{11} = 0,$$

$$\varphi_6 + \varphi_9 - \varphi_{12} = 0.$$

One independent solution is $\varphi_6 = \varphi_7 = \varphi_{10} = \varphi_{12} = 1$ and $\varphi_8 = -1$ and $\varphi_9 = \varphi_{11} = 0$. So

$$\mathbf{b} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

The rank of $[A^{5 \times 5}|\mathbf{b}]$ is four. So by the Rouché-Capelli theorem the cluster kernel in figure 18 has an infinite number of solutions and since the rank of the coefficient matrix is one less than its size, one extra invariant. The null space for figure 18 is:

$$\Phi = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, both kernels have an additional invariant each. Therefore, the original 104-triad cluster has two extra invariants, $J = N - M + 2 = 178 - 104 + 2 = 76$.